

# DEPTH THREE TOWERS AND JACOBSON-BOURBAKI CORRESPONDENCE

LARS KADISON

**ABSTRACT.** We introduce a notion of depth three tower of three rings  $C \subseteq B \subseteq A$  as a useful generalization of depth two ring extension. If  $A = \text{End } B_C$  and  $B|C$  is a Frobenius extension, this also captures the notion of depth three for a Frobenius extension in [10, 11] such that if  $B|C$  is depth three, then  $A|C$  is depth two (cf. [18]). If  $A$ ,  $B$  and  $C$  correspond to a tower of subgroups  $G > H > K$  via the group algebra over a fixed base ring, the depth three condition is the condition that subgroup  $K$  has normal closure  $K^G$  contained in  $H$ . For a depth three tower of rings, there is a pre-Galois theory for the ring  $\text{End } {}_B A_C$  and coring  $(A \otimes_B A)^C$  involving Morita context bimodules and left coideal subrings. This is applied in the last two sections to a specialization of a Jacobson-Bourbaki correspondence theorem for augmented rings to depth two extensions with depth three intermediate division rings.

## 1. INTRODUCTION

To a depth two extension  $A \supseteq C$  is associated a bialgebroid  $S = \text{End } {}_C A_C$  over the centralizer  $V$ , where  $S$  acts naturally on  $A$  to produce an intermediate ring of invariants  $A^S$  between  $C \subseteq A$ . The poset map  $A \supseteq B \supseteq C$  of D2 balanced subextensions into sub-bialgebroids  $\mathcal{S} = \text{End } {}_B A_B \subseteq \text{End } {}_C A_C$ , together with the poset map  $\mathcal{S} \rightsquigarrow A^S$  form a surjective Galois connection [1]. For example, if  $A$  is a simple algebra over a field  $C$ , the Jacobson-Bourbaki correspondence between intermediate fields  $C \subseteq B \subseteq Z(A)$  and subalgebras  $\mathcal{R}$  of the linear endomorphism algebra  $E = \text{End } A_C$  containing  $A^e$  is given by the Galois correspondence  $B \rightsquigarrow \text{End } A_B$  with inverse  $\mathcal{R} \rightsquigarrow \text{End } {}_{\mathcal{R}} A$ . The departure point of this paper is that the Jacobson-Bourbaki correspondence coincides with the depth two Galois connection, since finite dimensional algebras are D2,  $A^S \cong \text{End } {}_E A$  (see Theorem 5.3 and its corollary) and  $E \cong A \otimes_V S$  (see Theorem 5.1 and its corollary [11, Prop. 3.10]). In section 6 we reformulate various classical Jacobson-Bourbaki theorems for a field extension [7], separable field extension [22] and simple algebra over a field [19] in terms of bialgebroids, weak Hopf algebras (this case was considered in [22]), Hopf algebroids and their subobjects.

When  $C$  is not in the center of  $A$ , as in the case of the Jacobson-Bourbaki theorem for division rings [7, 21], then  $\text{End } {}_C A_C$  is a proper subring of  $\text{End } A_C$ , and the depth two Galois connection no longer coincides with the Jacobson-Bourbaki correspondence (which formally remains the same as above). To make headway here, we introduce a notion of depth three tower of rings (algebras or groups)  $A|B|C$ , which in case  $B = C$  is a depth two extension  $A|C$  or, in case  $A = \text{End } B_C$  and  $B|C$  is a free Frobenius extension, is depth three as defined in [10, 3.1]. Now

---

1991 *Mathematics Subject Classification.* 13B05, 16W30, 46L37, 81R15.

My thanks to David Harbater and the Penn Galois seminar for the invitation.

let  $B$  and  $C$  be division rings and  $A$  have an augmentation map to a division ring. We show in Theorem 6.3 that the depth three intermediate rings  $B$  of a D2 extension  $A \supseteq C$  are in Galois correspondence with coideal subrings  $\mathcal{R}$  of the bialgebroid  $\text{End}_C A_C$  that are finite projective over  $V$  such that  $\mathcal{R}A$  is simple. This correspondence factors through a generalized Jacobson-Bourbaki correspondence.

The paper is organized as follows. In section 2 we note that right or left D3 ring towers are characterized in terms either of the tensor-square, H-equivalent modules, quasibases or the endomorphism ring. We prove a Theorem 2.5 that a depth three Frobenius extension  $B|C$  embeds in a depth two extension  $A|C$  (where  $A = \text{End } B_C$ ). In section 3 we show that a tower of subgroups  $G > H > K$  of finite index with the condition that the normal closure  $K^G < H$  ensures that the group algebras  $F[G] \supseteq F[H] \supseteq F[K]$  are a depth three tower w.r.t. any base ring  $F$ . We propose that the converse is true if  $G$  is a finite group and  $F = \mathbb{C}$ . In section 4 we study the right coideal subring  $E = \text{End}_B A_C$  as well as the bimodule and co-ring  $P = (A \otimes_B A)^C$ , which provide the quasibases for a right D3 tower  $A|B|C$ . We show that right depth three towers may be characterized by  $P$  being finite projective as a left module over the centralizer  $V = A^C$  and a pre-Galois isomorphism  $A \otimes_B A \xrightarrow{\cong} A \otimes_V P$ .

In section 5 we study further Galois properties of D3 towers, such as the smash product decomposition of one of the endomorphism rings and the invariants as a bicommutator. In section 6, we generalize the Jacobson-Bourbaki correspondence, which associates  $\text{End } E_F$  to subfields  $F$  of  $E$  (or skew fields), and conversely associates  $\text{End } \mathcal{R}E$  to closed subrings  $\mathcal{R} \subseteq \text{End } E_F$ . We then compose this correspondence with an anti-Galois correspondence to prove the main Theorem 6.3: viz., there is a Galois correspondence between D3 intermediate division rings of a D2 extension of an augmented ring  $A$  over a division ring  $C$ , on the one hand, with Galois left coideal subrings of the bialgebroid  $\text{End}_C A_C$ , on the other hand. In Section 7, we apply Jacobson-Bourbaki correspondence to show that the Galois connection for separable field extensions in [22] is a Galois correspondence between weak Hopf subalgebras and intermediate fields.

**1.1. Historical remarks.** The notion of depth in the classification of subfactors describes where in the derived tower of centralizers, if at all, there occurs three successive algebras forming a basic construction  $C \hookrightarrow B \hookrightarrow \text{End } B_C$ . Depth two plays the most important role in finite depth classification theory [18]. This is partly because a finite depth subfactor embeds via its Jones tower into a depth two subfactor (see Theorem 2.5 for the depth three algebraic version). A subfactor  $B \subseteq A$  is depth two then if the centralizers  $V_A(B) \hookrightarrow V_{A_1}(B) \hookrightarrow V_{A_2}(B)$  is a basic construction, where  $A \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow A_3$  is a Jones tower of iterated basic constructions. The subfactor  $B \subseteq A$  is depth three if instead the centralizers  $V_{A_1}(B) \hookrightarrow V_{A_2}(B) \hookrightarrow V_{A_3}(B)$  is a basic construction. The algebraic property of finite depth may be described most easily starting with a Frobenius extension  $A \supseteq B$ , where the definition guarantees the existence of a bimodule homomorphism  $A \rightarrow B$  with dual bases for the finitely generated projective  $B$ -module  $A$  [10].

A careful algebraic study of the depth two condition on subalgebra  $B \subseteq A$  shows that it is most simply expressed as a type of central projectivity condition on the tensor-square  $A \otimes_B A$  w.r.t.  $A$  as natural  $A$ - $B$ -bimodules and  $B$ - $A$ -bimodules [11]. There is a Galois theory connected to this viewpoint with Galois quantum

groupoids, in the category of Hopf algebroids [11, 22, 12]. Although a future view-point on depth two ring extension in this generality might be that it is better called a “normal extension,” there are still some outstanding problems (e.g., are D2 Hopf subalgebras normal?). “Depth two” does presently suggest that it is part of a larger theory of depth 2, 3 and beyond for ring extensions. Indeed depth three does lend itself, after reformulation, to a notion for ring extensions as in the preprint to [11].

In this paper we prefer to view depth three as a property most naturally associated to a tower of three algebras or rings  $C \subseteq B \subseteq A$ . This tower is right depth three (rD3) if  $A \otimes_B A$  is  $A$ - $C$ -isomorphic to a direct summand of  $A \oplus \cdots \oplus A$ . The advantage of this definition over the one in [11, preprint version] is that it is close to the depth two definition so that a substantial amount of depth two theory is available as we see in this paper. At the same time, we show in the last two sections that depth three towers plays a role in Galois correspondence theory for depth two extensions. The relation of depth three towers with classical depth three subfactors may be seen as follows: if  $C \subseteq B$  is a Frobenius extension with  $A = \text{End } B_C$ , it follows that  $A \cong B \otimes_C B$ , that  ${}_A A \otimes_B A_C$  reduces to  ${}_A B \otimes_C B \otimes_C B_C$  and  ${}_A A_C$  to  ${}_A B \otimes_C B_C$ , the terms in which the depth three condition is expressed in [11, preprint version].

## 2. DEFINITION AND FIRST PROPERTIES OF DEPTH THREE TOWERS

Let  $A$ ,  $B$  and  $C$  denote rings with identity element, and  $C \rightarrow B$ ,  $B \rightarrow A$  denote ring homomorphisms preserving the identities. We use ring extension notation  $A|B|C$  for  $C \rightarrow B \rightarrow A$  and call this a tower of rings: an important special case if of course  $C \subseteq B \subseteq A$  of subrings  $B$  in  $A$  and  $C$  in  $B$ . Of most importance to us are the induced bimodules such as  ${}_B A_C$  and  ${}_C A_B$ . We may naturally also choose to work with algebras over commutative rings, and obtain almost identical results.

We denote the centralizer subgroup of a ring  $A$  in an  $A$ - $A$ -bimodule  $M$  by  $M^A = \{m \in M \mid \forall a \in A, ma = am\}$ . We also use the notation  $V_A(C) = A^C$  for the centralizer subring of  $C$  in  $A$ . This should not be confused with our notation  $K^G$  for the normal closure of a subgroup  $K < G$ . Notation like  $\text{End } B_C$  will denote the ring of endomorphisms of the module  $B_C$  under composition and addition. We let  $N_R^n$  denote the  $n$ -fold direct sum of a right  $R$ -module  $N$  with itself; let  $M_R \oplus * \cong N_R^n$  denote the module  $M$  is isomorphic to a direct summand of  $N_R^n$ .

**Definition 2.1.** *A tower of rings  $A|B|C$  is right depth three (rD3) if the tensor-square  $A \otimes_B A$  is isomorphic as  $A$ - $C$ -bimodules to a direct summand of a finite direct sum of  $A$  with itself: in module-theoretic symbols, this becomes, for some positive integer  $N$ ,*

$$(1) \quad {}_A A \otimes_B A_C \oplus * \cong {}_A A_C^N$$

By switching to  $C$ - $A$ -bimodules instead, we similarly define a *left D3 tower* of rings. The theory for these is dual to that for rD3 towers; we briefly consider it at the end of this section. As an alternative to referring to a rD3 tower  $A|B|C$ , we may refer to  $B$  as an rD3 *intermediate ring* of  $A|C$ , if  $C \rightarrow A$  factors through  $B \rightarrow A$  and  $A|B|C$  is rD3.

Recall that over a ring  $R$ , two modules  $M_R$  and  $N_R$  are H-equivalent if  $M_R \oplus * \cong N_R^n$  and  $N_R \oplus * \cong M_R^m$  for some positive integers  $n$  and  $m$ . In this case, the endomorphism rings  $\text{End } M_R$  and  $\text{End } N_R$  are Morita equivalent with context bimodules  $\text{Hom}(M_R, N_R)$  and  $\text{Hom}(N_R, M_R)$ .

**Lemma 2.2.** *A tower  $A|B|C$  of rings is rD3 iff the natural  $A$ - $C$ -bimodules  $A \otimes_B A$  and  $A$  are  $H$ -equivalent.*

*Proof.* We note that for any tower of rings,  $A \oplus * \cong A \otimes_B A$  as  $A$ - $C$ -bimodules, since the epi  $\mu : A \otimes_B A \rightarrow A$  splits as an  $A$ - $C$ -bimodule arrow.  $\square$

Since for any tower of rings  $\text{End}_A A_C$  is isomorphic to the centralizer  $V_A(C) = A^C$  (or anti-isomorphic according to convention), we see from the lemma that the notion of rD3 has something to do with classical depth three. Indeed,

**Example 2.3.** If  $B|C$  is a Frobenius extension, with Frobenius system  $(E, x_i, y_i)$  satisfying for each  $a \in A$ ,

$$(2) \quad \sum_i E(ax_i)y_i = a = \sum_i x_i E(y_i a)$$

then  $B \otimes_C B \cong \text{End } B_C := A$  via  $x \otimes_B y \mapsto \lambda_x \circ E \circ \lambda_y$  for left multiplication  $\lambda_x$  by element  $x \in B$ . Let  $B \rightarrow A$  be this mapping  $B \hookrightarrow \text{End } B_C$  given by  $b \mapsto \lambda_b$ . It is then easy to show that  ${}_A B \otimes_C B \otimes_C B_C \cong {}_A A \otimes_B A_C$ , so that for Frobenius extensions, condition (1) is equivalent to the condition for rD3 in preprint [11], which in turn slightly generalizes the condition in [10] for depth three free Frobenius extension. We should make note here that right or left depth three would be equivalent notions for Frobenius extensions, since  $\text{End } B_C$  and  $\text{End } {}_C B$  are anti-isomorphic for such.

Another litmus test for a correct notion of depth three is that depth two extensions should be depth three in a certain sense. Recall that a ring extension  $A|B$  is right depth two (rD2) if the tensor-square  $A \otimes_B A$  is  $A$ - $B$ -bimodule isomorphic to  $N$  copies of  $A$  in a direct sum with itself:

$$(3) \quad {}_A A \otimes_B A_B \oplus * \cong {}_A A_B^N$$

Since the notions pass from ring extension to tower of rings, there are several cases to look at.

**Proposition 2.4.** *Suppose  $A|B|C$  is a tower of rings. We note:*

- (1) *If  $B = C$  and  $B \rightarrow C$  is the identity mapping, then  $A|B|C$  is rD3  $\Leftrightarrow A|B$  is rD2.*
- (2) *If  $A|B$  is rD2, then  $A|B|C$  is rD3 w.r.t. any ring extension  $B|C$ .*
- (3) *If  $A|C$  is rD2 and  $B|C$  is a separable extension, then  $A|B|C$  is rD3.*
- (4) *If  $B|C$  is left D2, and  $A = \text{End } B_C$ , then  $A|B|C$  is left D3.*
- (5) *If  $C$  is the trivial subring, any ring extension  $A|B$ , where  ${}_B A$  is finite projective, together with  $C$  is rD3.*

*Proof.* The proof follows from comparing eqs. (1) and (3), noting that  $A \otimes_B A \oplus * \cong A \otimes_C A$  as natural  $A$ - $A$ -bimodules if  $B|C$  is a separable extension (thus having a separability element  $e = e^1 \otimes_C e^2 \in (B \otimes_C B)^B$  satisfying  $e^1 e^2 = 1$ ), and finally from [12] that  $B|C$  left D2 extension  $\Rightarrow A|B$  is left D2 extension if  $A = \text{End } B_C$ . The last statement follows from tensoring  ${}_B A \oplus * \cong {}_B B^n$  by  ${}_A A \otimes_B -$ .  $\square$

The next theorem is a converse and algebraic simplification of a key fact in subfactor Galois theory (the  $n = 3$  case): a depth three subfactor  $N \subseteq M$  yields a depth two subfactor  $N \subseteq M_1$ , w.r.t. its basic constuction  $M_1 \cong M \otimes_N M$ . In preparation, let us call a ring extension  $B|C$  rD3 if the endomorphism ring tower

$A|B|C$  is rD3, where  $A = \text{End } B_C$  and  $A|B$  has underlying map  $\lambda : B \rightarrow \text{End } B_C$ , the left regular mapping given by  $\lambda(x)(b) = xb$  for all  $x, b \in B$ .

**Theorem 2.5.** *Suppose  $B|C$  is a Frobenius extension and  $A = \text{End } B_C$ . If  $B|C$  is rD3, then the composite extension  $A|C$  is D2.*

*Proof.* There is a well-known bimodule isomorphism for a Frobenius extension  $B|C$ , between its endomorphism ring and its tensor-square,  ${}_B A_B \cong {}_B B \otimes_C B_B$ . Tensoring by  ${}_A A \otimes_B - \otimes_B A_A$ , we obtain  $A \otimes_C A \cong A \otimes_B A \otimes_B A$  as natural  $A$ - $A$ -bimodules. Now restrict the bimodule isomorphism in eq. (1) on the right to  $B$ -modules and tensor by  ${}_A A \otimes_B -$  to obtain  ${}_A A \otimes_C A_C \oplus * \cong {}_A A \otimes_B A_C^N$  after substitution of the tensor-cube over  $B$  by the tensor-square over  $C$ . By another application of eq. (1) we arrive at

$${}_A A \otimes_C A_C \oplus * \cong {}_A A_C^{N^2}$$

Thus  $A|C$  is right D2. Since it is a Frobenius extension as well, it is also left depth two.  $\square$

We introduce quasi-bases for right depth three towers.

**Theorem 2.6.** *A tower  $A|B|C$  is right depth three iff there are  $N$  elements each of  $\gamma_i \in \text{End } {}_B A_C$  and of  $u_i \in (A \otimes_B A)^C$  satisfying (for each  $x, y \in A$ )*

$$(4) \quad x \otimes_B y = \sum_{i=1}^N x \gamma_i(y) u_i$$

*Proof.* From the condition (1), there are obviously  $N$  maps each of

$$(5) \quad f_i \in \text{Hom}({}_A A_C, {}_A A \otimes_B A_C), \quad g_i \in \text{Hom}({}_A A \otimes_B A_C, {}_A A_C)$$

such that  $\sum_{i=1}^N f_i \circ g_i = \text{id}_{{}_A A \otimes_B A_C}$ . First, we note that for any tower of rings, not necessarily rD3,

$$(6) \quad \text{Hom}({}_A A_C, {}_A A \otimes_B A_C) \cong (A \otimes_B A)^C$$

via  $f \mapsto f(1_A)$ . The inverse is given by  $p \mapsto ap$  where  $p = p^1 \otimes_B p^2 \in (A \otimes_B A)^C$  using a Sweedler-type notation that suppresses a possible summation over simple tensors.

The other hom-group above also has a simplification. We note that for any tower,

$$(7) \quad \text{Hom}({}_A A \otimes_B A_C, {}_A A_C) \cong \text{End } {}_B A_C$$

via  $F \mapsto F(1_A \otimes_B -)$ . Given  $\alpha \in \text{End } {}_B A_C$ , we define an inverse sending  $\alpha$  to the homomorphism  $x \otimes_B y \mapsto x \alpha(y)$ .

Let  $f_i$  correspond to  $u_i \in (A \otimes_B A)^C$  and  $g_i$  correspond to  $\gamma_i \in \text{End } {}_B A_C$  via the mappings just described. We compute:

$$x \otimes_B y = \sum_i f_i(g_i(x \otimes_B y)) = \sum_i f_i(x \gamma_i(y)) = \sum_i x \gamma_i(y) u_i,$$

which establishes the rD3 quasibases equation in the theorem, given an rD3 tower.

For the converse, suppose we have  $u_i \in (A \otimes_B A)^C$  and  $\gamma_i \in \text{End } {}_B A_C$  satisfying the equation in the theorem. Then map  $\pi : A^N \rightarrow A \otimes_B A$  by

$$\pi : (a_1, \dots, a_N) \mapsto \sum_i a_i u_i,$$

an  $A$ - $C$ -bimodule epimorphism split by the mapping  $\sigma : A \otimes_B A \hookrightarrow A^N$  given by

$$\sigma(x \otimes_B y) := (x\gamma_1(y), \dots, x\gamma_N(y)).$$

It follows from the equation above that  $\pi \circ \sigma = \text{id}_{A \otimes_B A}$ .  $\square$

**2.1. Left D3 towers and quasibases.** A tower of rings  $A|B|C$  is left D3 if the tensor-square  $A \otimes_B A$  is an  $C$ - $A$ -bimodule direct summand of  $A^N$  for some  $N$ . If  $B = C$ , this recovers the definition of a left depth two extension  $A|B$ . There is a left version of all results in this paper: we note that  $A|B|C$  is a right D3 tower if and only if  $A^{\text{op}}|B^{\text{op}}|C^{\text{op}}$  is a left D3 tower (cf. [12]).

The next theorem refers to notation established in the example above.

**Theorem 2.7.** *Suppose  $B|C$  is a Frobenius extension with  $A = \text{End } B_C$ . Then  $A|B|C$  is right depth three if and only if  $A|B|C$  is left depth three.*

*Proof.* It is well-known that also  $A|B$  is a Frobenius extension. Then  $A \otimes_B A \cong \text{End } A_B$  as natural  $A$ - $A$ -bimodules. Also  $A \otimes_B A \cong \text{End } _B A$  by a similar mapping utilizing the Frobenius homomorphism in one direction, and the dual bases in the other. Composing gives us an anti-isomorphism of the left and right endomorphism rings denoted by  $f \mapsto f^\tau$ .

Now note the following characterization of left D3 with proof almost identical with that of [14, Prop. 3.8]: If  $A|B|C$  is a tower where  $A_B$  is finite projective, then  $A|B|C$  is left D3  $\Leftrightarrow \text{End } A_B \oplus * \cong A^N$  as natural  $A$ - $C$ -bimodules. The proof involves noting that  $\text{End } A_B \cong \text{Hom}(A \otimes_B A_A, A_A)$  as natural  $A$ - $C$ -bimodules via

$$f \mapsto (a \otimes a' \mapsto f(a)a').$$

The finite projectivity is used for reflexivity in hom'ing this isomorphism, thus proving the converse statement.

Similarly, if  $A|B|C$  is a tower where  $_B A$  is finite projective, then  $A|B|C$  is right D3 if and only if  $\text{End } _B A \oplus * \cong A^N$  as natural  $C$ - $A$ -bimodules.

Of course a Frobenius extension satisfies both finite projectivity conditions. The anti-isomorphism of the left and right endomorphism rings twists the  $C$ - $A$ -structure of  $\text{End } _B A$  given by  $\rho_c \circ f \circ \rho_a$  to the  $A$ - $C$ -structure on  $\text{End } A_B$  given by  $\lambda_a \circ f^\tau \circ \lambda_c$ , thereby demonstrating the equivalence of left and right D3 conditions on  $A \otimes_B A$  relative to  $A \cong \text{End } A_A$ .  $\square$

In a fairly obvious reversal to opposite ring structures in the proof of Theorem 2.6, we see that a tower  $A|B|C$  is left D3 iff there are  $N$  elements  $\beta_j \in \text{End } _C A_B$  and  $N$  elements  $t_j \in (A \otimes_B A)^C$  such that for all  $x, y \in A$ , we have

$$(8) \quad x \otimes_B y = \sum_{j=1}^N t_j \beta_j(x) y$$

We record the characterization of left D3, noted above in the proof, for towers satisfying a finite projectivity condition.

**Theorem 2.8.** *Suppose  $A|B|C$  is a tower of rings where  $A_B$  is finite projective. Then this tower is left D3 if and only if the natural  $A$ - $C$ -bimodules satisfy for some  $N$ ,*

$$(9) \quad \text{End } A_B \oplus * \cong A^N$$

Finally we define a tower  $A|B|C$  to be D3 if it is both left D3 and right D3.

## 3. DEPTH THREE FOR TOWERS OF GROUPS

Fix a base ring  $F$ . Groups give rise to rings via  $G \mapsto F[G]$ , the functor associating the group algebra  $F[G]$  to a group  $G$ . Therefore we can pull back the notion of depth 2 or 3 for ring extensions or towers to the category of groups (so long as reference is made to the base ring).

In the paper [9], a depth two subgroup w.r.t. the complex numbers is shown to be equivalent to the notion of normal subgroup for finite groups. This consists of two results. The easier result is that over any base ring, a normal subgroup of finite index is depth two by exhibiting left or right D2 quasibases via coset representatives and projection onto cosets. This proof suggests that the converse hold as well. The second result is a converse for complex finite dimensional D2 group algebras where normality of the subgroup is established using character theory and Mackey's subgroup theorem.

In this section, we will similarly do the first step in showing what group-theoretic notion corresponds to depth three tower of rings. Let  $G > H > K$  be a tower of groups, where  $G$  is a finite group,  $H$  is a subgroup, and  $K$  is a subgroup of  $H$ . Let  $A = F[G]$ ,  $B = F[H]$  and  $C = F[K]$ . Then  $A|B|C$  is a tower of rings, and we may ask what group-theoretic notion on  $G > H > K$  will guarantee, with fewest possible hypotheses, that  $A|B|C$  is rD3.

**Theorem 3.1.** *The tower of groups algebras  $A|B|C$  is D3 if the corresponding tower of groups  $G > H > K$  satisfies*

$$(10) \quad K^G < H$$

where  $K^G$  denotes the normal closure of  $K$  in  $G$ .

*Proof.* Let  $\{g_1, \dots, g_N\}$  be double coset representatives such that  $G = \coprod_{i=1}^N Hg_iK$ . Define  $\gamma_i(g) = 0$  if  $g \notin Hg_iK$  and  $\gamma_i(g) = g$  if  $g \in Hg_iK$ . Of course,  $\gamma_i \in \text{End}_{BA} C$  for  $i = 1, \dots, N$ .

Since  $K^G \subseteq H$ , we have  $gK \subseteq Hg$  for each  $g \in G$ . Hence for each  $k \in K$ ,  $g_jk = hg_j$  for some  $h \in H$ . It follows that

$$g_j^{-1} \otimes_B g_jk = g_j^{-1}h \otimes_B g_j = kg_j^{-1} \otimes_B g_j.$$

Given  $g \in G$ , we have  $g = hg_jk$  for some  $j = 1, \dots, N$ ,  $h \in H$ , and  $k \in K$ . Then we compute:

$$1 \otimes_B g = 1 \otimes_B hg_jk = hg_jg_j^{-1} \otimes_B g_jk = hg_jkg_j^{-1} \otimes_B g_j$$

so  $1 \otimes_B g = \sum_i \gamma_i(g)g_i^{-1} \otimes_B g_i$  where  $g_i^{-1} \otimes_B g_i \in (A \otimes_B A)^C$ . By theorem then,  $A|B|C$  is an rD3 tower.

The proof that the tower of group algebras is left D3 is entirely symmetrical via the inverse mapping.  $\square$

The theorem is also valid for infinite groups where the index  $[G : H]$  is finite, since  $HgK = Hg$  for each  $g \in G$ .

Notice how the equivalent notions of depth two and normality for finite groups over  $\mathbb{C}$  yields the Proposition 2.4 for groups. Suppose we have a tower of groups  $G > H > K$  where  $K^G \subseteq H$ . If  $K = H$ , then  $H$  is normal (D2) in  $G$ . If  $K = \{e\}$ , then it is rD3 together with any subgroup  $H < G$ . If  $H \triangleleft G$  is a normal subgroup, then necessarily  $K^G \subseteq H$ . If  $K \triangleleft G$ , then  $K^G = K < H$  and the tower is D3.

Question: Can the character-theoretic proof in [9] be adapted to prove that a D3 tower  $\mathbb{C}[G] \supseteq \mathbb{C}[H] \supseteq \mathbb{C}[K]$  where  $G$  is a finite group satisfies  $K^G < H$ ?

#### 4. ALGEBRAIC STRUCTURE ON $\text{End}_B A_C$ AND $(A \otimes_B A)^C$

In this section, we study the calculus of some structures definable for an rD3 tower  $A|B|C$ , which reduce to the dual bialgebroids over the centralizer of a ring extension in case  $B = C$  and their actions/coactions. Throughout the section,  $A|B|C$  will denote a right depth three tower of rings,

$$P := (A \otimes_B A)^C, \quad Q := (A \otimes_C A)^B,$$

which are bimodules with respect to the two rings familiar from depth two theory,

$$T := (A \otimes_B A)^B, \quad U := (A \otimes_C A)^C$$

Note that  $P$  and  $Q$  are isomorphic to two  $A$ - $A$ -bimodule Hom-groups:

$$(11) \quad P \cong \text{Hom}(A \otimes_C A, A \otimes_B A), \quad Q \cong \text{Hom}(A \otimes_B A, A \otimes_C A).$$

Recall that  $T$  and  $U$  have multiplications given by

$$tt' = t'^1 t^1 \otimes_B t^2 t'^2, \quad uu' = u'^1 u^1 \otimes_C u^2 u'^2,$$

where  $1_T = 1_A \otimes 1_A$  and a similar expression for  $1_U$ . Namely, the bimodule  ${}_T P_U$  is given by

$$(12) \quad {}_T P_U : t \cdot p \cdot u = u^1 p^1 t^1 \otimes_B t^2 p^2 u^2$$

The bimodule  ${}_U Q_T$  is given by

$$(13) \quad {}_U Q_T : u \cdot q \cdot t = t^1 q^1 u^1 \otimes_C u^2 q^2 t^2$$

We have the following result, also mentioned in passing in [13] with several additional hypotheses.

**Proposition 4.1.** *The bimodules  $P$  and  $Q$  over the rings  $T$  and  $U$  form a Morita context with associative multiplications*

$$(14) \quad P \otimes_U Q \rightarrow T, \quad p \otimes q \mapsto pq = q^1 p^1 \otimes_B p^2 q^2$$

$$(15) \quad Q \otimes_T P \rightarrow U, \quad q \otimes p \mapsto qp = p^1 q^1 \otimes_C q^2 p^2$$

*If  $B|C$  is an H-separable extension, then  $T$  and  $U$  are Morita equivalent rings via this context.*

*Proof.* The equations  $p(qp') = (pq)p'$  and  $q(pq') = (qp)q'$  for  $p, p' \in P$  and  $q, q' \in Q$  follow from the four equations directly above.

Note that

$$T \cong \text{End}_A A \otimes_B A_A, \quad U \cong \text{End}_A A \otimes_C A_A$$

as rings. We now claim that the hypotheses on  $A|B$ ,  $A|C$  and  $B|C$  imply that the  $A$ - $A$ -bimodules  $A \otimes_B A$  and  $A \otimes_C A$  are H-equivalent. Then the endomorphism rings above are Morita equivalent via context bimodules given by eqs. (11), which proves the proposition.

Since  $B|C$  is H-separable, it is in particular separable, and the canonical  $A$ - $A$ -epi  $A \otimes_C A \rightarrow A \otimes_B A$  splits via an application of a separability element. Thus,  $A \otimes_B A \oplus * \cong A \otimes_C A$ . Also,  $B \otimes_C B \oplus * \cong B^N$  as  $B$ - $B$ -bimodules for some positive integer  $N$ . Therefore,  $A \otimes_C A \oplus * \cong A \otimes_B A^N$  as  $A$ - $A$ -bimodules by an application of the functor  $A \otimes_B ? \otimes_B A$ . Hence,  $A \otimes_B A$  and  $A \otimes_C A$  are H-equivalent  $A^e$ -modules (i.e.,  $A$ - $A$ -bimodules).  $\square$



We denote the centralizer subrings  $A^C$  and  $A^B$  of  $A$  by

$$(16) \quad R := V_A(B) \subseteq V_A(C) := V$$

We have generalized anchor mappings [13],

$$(17) \quad R \otimes_T P \longrightarrow V, \quad r \otimes p \longmapsto p^1 r p^2$$

$$(18) \quad V \otimes_U Q \longrightarrow R, \quad v \otimes q \longmapsto q^1 v q^2$$

**Proposition 4.2.** *The two generalized anchor mappings are bijective if  $B|C$  is  $H$ -separable.*

*Proof.* Denote  $r \cdot p := p^1 r p^2$  and  $v \cdot q := q^1 v q^2$ . From the previous proposition, there are elements  $p_i \in P$  and  $q_i \in Q$  such that  $\sum_i p_i q_i = 1_T$ ; in addition,  $p'_j \in P$  and  $q'_j \in Q$  such that  $1_U = \sum_j q'_j p'_j$ . Let  $v \in V$ , then

$$v = v \cdot 1_U = \sum_j v \cdot (q'_j p'_j) = \sum_j (v \cdot q'_j) \cdot p'_j$$

and a similar computation starting with  $r = r \cdot 1_T$  shows that the two generalized anchor mappings are surjective.

In general, we have the corestriction of the inclusion  $T \subseteq A \otimes_B A$ ,

$$(19) \quad {}_T T \hookrightarrow {}_T P$$

which is split as a left  $T$ -module monic by  $p \mapsto e^1 p e^2$  in case there is a separability element  $e = e^1 \otimes_C e^2 \in B \otimes_C B$ . Similarly,

$$(20) \quad {}_U Q \hookrightarrow {}_U U$$

is a split monic in case  $B|C$  is separable. Of course, if  $B|C$  is  $H$ -separable, we note from Proposition 4.1 and Morita theory that  $P$  and  $Q$  are projective generators on both sides.

It follows from faithful flatness that the anchor mappings are also injective.  $\square$

Note that  $P$  is a  $V$ - $V$ -bimodules (via the commuting homomorphism and anti-homomorphism  $V \rightarrow U \leftarrow V$ ):

$$(21) \quad {}_V P_V : \quad v \cdot p \cdot v' = v p^1 \otimes_B p^2 v'$$

Note too that  $E = \text{End}_B A_C$  is an  $R$ - $V$ -bimodule via

$$(22) \quad {}_R E_V : \quad r \cdot \alpha \cdot v = r \alpha(-) v$$

Note the subring and over-ring

$$(23) \quad \text{End}_B A_B \subseteq E \subseteq \text{End}_C A_C$$

which are the total algebras of the left  $R$ - and  $V$ - bialgebroids in depth two theory [11, 12, 13].

**Lemma 4.3.** *The modules  ${}_V P$  and  $E_V$  are finitely generated projective. In case  $A|C$  is left  $D2$ , the subring  $E$  is a right coideal subring of the left  $V$ -bialgebroid  $\text{End}_C A_C$ .*

*Proof.* This follows from eq. (4), since  $p \in P \subseteq A \otimes_B A$ , so

$$p = \sum_i p^1 \gamma_i(p^2) u_i$$

where  $u_i \in P$  and  $p \mapsto p^1 \gamma_i(p^2)$  is in  $\text{Hom}({}_V P, {}_V V)$ , thus dual bases for a finite projective module. The second claim follows similarly from

$$\alpha = \sum_i \gamma_i(-) u_i^1 \alpha(u^2)$$

where  $\gamma_i \in E$  and  $\alpha \mapsto u^1 \alpha(u^2)$  are mappings in  $\text{Hom}(E_V, V_V)$ .

Now suppose  $\beta_j \in S := \text{End}_C A_C$  and  $t_j \in (A \otimes_C A)^C$  are left D2 quasibases of  $A|C$ . Recall that the coproduct  $\Delta : S \rightarrow S \otimes_V S$  given by  $(\beta \in S)$

$$(24) \quad \Delta(\beta) = \sum_j \beta(-t_j^1) t_j^2 \otimes_V \beta_j$$

makes  $S$  a left  $V$ -bialgebroid [11]. Of course this restricts and corestricts to  $\alpha \in E$  as follows:  $\Delta(\alpha) \in E \otimes_V S$ . Hence,  $E$  is a right coideal subring of  $S$ .  $\square$

In fact, if  $A|B$  is also D2, and  $\mathcal{S} = \text{End}_B A_B$ , then  $E$  is similarly shown to be an  $\mathcal{S}$ - $\mathcal{S}$ -bicomodule ring. For we recall the coaction  $E \rightarrow \mathcal{S} \otimes_R E$  given by

$$(25) \quad \alpha_{(-1)} \otimes_R \alpha_{(0)} = \sum_i \tilde{\gamma}_i \otimes \tilde{u}_i^1 \alpha(\tilde{u}_i^2 -)$$

where  $\tilde{\gamma}_i \in \mathcal{S}$  and  $\tilde{u}_i \in (A \otimes_B A)^B$  are right D2 quasibases of  $A|B$  (restriction of [12, eq. (19)]).

Twice above we made use of a  $V$ -bilinear pairing  $P \otimes E \rightarrow V$  given by

$$(26) \quad \langle p, \alpha \rangle := p^1 \alpha(p^2), \quad (p \in P = (A \otimes_B A)^C, \alpha \in E = \text{End}_B A_C)$$

**Lemma 4.4.** *The pairing above is nondegenerate. It induces  $E_V \cong \text{Hom}({}_V P, {}_V V)$  via  $\alpha \mapsto \langle -, \alpha \rangle$ .*

*Proof.* The mapping has the inverse  $F \mapsto \sum_i \gamma_i(-) F(u_i)$  where  $\gamma_i \in E, u_i \in P$  are rD3 quasibases for  $A|B|C$ . Indeed,  $\sum_i \langle p, \gamma_i \rangle F(u_i) = F(\sum_i p^1 \gamma_i(p^2) u_i) = F(p)$  for each  $p \in P$  since  $F$  is left  $V$ -linear, and for each  $\alpha \in E$ , we note that  $\sum_i \gamma_i(-) \langle u_i, \alpha \rangle = \alpha$ .  $\square$

**Proposition 4.5.** *There is a  $V$ -coring structure on  $P$  left dual to the ring structure on  $E$ .*

*Proof.* We note that

$$(27) \quad P \otimes_V P \cong (A \otimes_B A \otimes_B A)^C$$

via  $p \otimes p' \mapsto p^1 \otimes p^2 p'^1 \otimes p'^2$  with inverse

$$p = p^1 \otimes p^2 \otimes p^3 \mapsto \sum_i (p^1 \otimes_B p^2 \gamma_i(p^3)) \otimes_V u_i.$$

Via this identification, define a  $V$ -linear coproduct  $\Delta : P \rightarrow P \otimes_V P$  by

$$(28) \quad \Delta(p) = p^1 \otimes_B 1_{A \otimes_B P} p^2.$$

Alternatively, using Sweedler notation and rD3 quasibases,

$$(29) \quad p_{(1)} \otimes_V p_{(2)} = \sum_i (p^1 \otimes_B \gamma_i(p^2)) \otimes_V u_i$$

Define a  $V$ -linear counit  $\varepsilon : P \rightarrow V$  by  $\varepsilon(p) = p^1 p^2$ . The counital equations follow readily [2].

Recall from Sweedler [21] that the  $V$ -coring  $(P, V, \Delta, \varepsilon)$  has left dual ring  ${}^*P := \text{Hom}({}_V P, {}_V V)$  given by Sweedler notation by

$$(30) \quad (f * g)(p) = f(p_{(1)} g(p_{(2)}))$$

with  $1 = \varepsilon$ . Let  $\alpha, \beta \in E$ . If  $f = \langle -, \alpha \rangle$  and  $g = \langle -, \beta \rangle$ , we compute  $f * g = \langle -, \alpha \circ \beta \rangle$  below, which verifies the claim:

$$f(p_{(1)} g(p_{(2)})) = \sum_i \langle p^1 \otimes_B \gamma_i(p^2) \langle u_i, \beta \rangle, \alpha \rangle = \langle p^1 \otimes_B \beta(p^2), \alpha \rangle = \langle p, \alpha \circ \beta \rangle.$$

□

In addition, we note that  $P$  is  $V$ -coring with grouplike element

$$(31) \quad g_P := 1_A \otimes_B 1_A$$

since  $\Delta(g_P) = 1 \otimes 1 \otimes 1 = g_P \otimes_V g_P$  and  $\varepsilon(g_P) = 1$ .

There is a pre-Galois structure on  $A$  given by the right  $P$ -comodule structure  $\delta : A \rightarrow A \otimes_V P$ ,  $\delta(a) = a_{(0)} \otimes_V a_{(1)}$  defined by

$$(32) \quad \delta(a) := \sum_i \gamma_i(a) \otimes_V u_i.$$

The pre-Galois isomorphism  $\beta : A \otimes_B A \xrightarrow{\cong} A \otimes_V P$  given by

$$(33) \quad \beta(a \otimes_B a') = a a'_{(0)} \otimes_V a'_{(1)}$$

is utilized below in another characterization of right depth three towers.

**Theorem 4.6.** *A tower of rings  $A|B|C$  is right depth three if and only if  ${}_V P$  is finite projective and  $A \otimes_V P \cong A \otimes_B A$  as natural  $A$ - $C$ -bimodules.*

*Proof.* ( $\Rightarrow$ ) If  ${}_V P \oplus * \cong {}_V V^N$  and  $A \otimes_V P \cong A \otimes_B A$ , then tensoring by  $A \otimes_V -$ , we obtain  $A \otimes_B A \oplus * \cong A^N$  as natural  $A$ - $C$ -bimodules, the rD3 defining condition on a tower.

( $\Leftarrow$ ) In Proposition we see that  ${}_V P$  is f.g. projective. Map  $A \otimes_V P \rightarrow A \otimes_B A$  by  $a \otimes p \mapsto a p^1 \otimes_B p^2$ , clearly an  $A$ - $C$ -bimodule homomorphism. The inverse is the “pre-Galois” isomorphism,

$$(34) \quad \beta : A \otimes_B A \rightarrow A \otimes_V P, \quad \beta(a \otimes_B a') = \sum_i a \gamma_i(a') \otimes_V u_i$$

since  $\sum_i a p^1 \gamma_i(p^2) \otimes_V u_i = a \otimes_V p$  and  $\sum_i a \gamma_i(a') u_i = a \otimes a'$  for  $a, a' \in A, p \in P$ . □

## 5. FURTHER GALOIS PROPERTIES OF DEPTH THREE

We will show here that the smaller of the endomorphism rings of a depth three tower decomposes tensorially over the overalgebra and the mixed bimodule endomorphism ring studied above. In case the composite ring extension is depth two, this is a smash product decomposition in terms of a coideal subring of a bialgebroid. Finally, we express the invariants of this coideal subring acting on the overalgebra in terms of a bicommutator.

**Theorem 5.1.** *If  $A|B|C$  is left D3, then*

$$(35) \quad \text{End } A_B \cong A \otimes_V \text{End } {}_C A_B$$

*via the homomorphism  $A \otimes_V \text{End } {}_C A_B \rightarrow \text{End } A_B$  given by  $a \otimes_V \alpha \mapsto \lambda_a \circ \alpha$ .*

*Proof.* Given a left D3 quasibase  $\beta_j \in \text{End}_C A_B$  and  $t_j \in (A \otimes_B A)^C$ , note that the mapping  $\text{End } A_B \rightarrow A \otimes_V \text{End}_C A_B$  given by

$$(36) \quad f \mapsto \sum_j f(t_j^1) t_j^2 \otimes_V \beta_j$$

is an inverse to the homomorphism above.  $\square$

**Corollary 5.2.** *If  $A|C$  is additionally D2, then  $\text{End}_C A_B$  a left coideal subring of  $\text{End}_C A_C$  and there is a ring isomorphism with a smash product ring,*

$$(37) \quad \text{End } A_B \cong A \rtimes \text{End}_C A_B$$

*Proof.* Recall from depth two theory [11] that the  $V$ -bialgebroid  $\text{End}_C A_C$  acts on the module algebra  $A$  by simple evaluation,  $\beta \triangleright a = \beta(a)$ . That the action is measuring is not hard to see from the formula for the coproduct on  $\text{End}_C A_C$  given by

$$(38) \quad \Delta(\beta) = \beta_{(1)} \otimes_V \beta_{(2)} := \sum_k \tilde{\gamma}_k \otimes_V \tilde{u}_k^1 \beta(\tilde{u}_k^2 -)$$

where  $\tilde{\gamma}_k \in \text{End}_C A_C$  and  $\tilde{u}_k \in (A \otimes_C A)^C$  are right D2 quasibase for the composite ring extension  $A|C$ . Note then that for  $\alpha \in \text{End}_C A_B \subseteq \text{End}_C A_C$ , the equation yields  $\alpha_{(1)} \otimes_V \alpha_{(2)} \in \text{End}_C A_C \otimes_V \text{End}_C A_B$ . Hence,  $\text{End}_C A_B$  is a left coideal subring. The details and verifications of the definition of such an object, over a smaller base ring than that of the bialgebroid, are rather straightforward and left to the reader.

As a consequence of the smash product formula  $\text{End } A_C \cong A \rtimes \text{End}_C A_C$  over the centralizer  $V$ , we restrict to  $\text{End } A_B \subseteq \text{End } A_C$ , apply the theorem above, to obtain the equation for  $\alpha, \beta \in \text{End}_C A_B$ ,

$$(39) \quad (a \# \alpha)(b \# \beta) = a(\alpha_{(1)} \triangleright b) \# \alpha_{(2)} \circ \beta \in A \otimes_V \text{End}_C A_B$$

where  $a, b \in A$ , and  $\rtimes, \#$  are used interchangeably.  $\square$

In case  $A|C$  continues to be a D2 extension, the theorem below will characterize the subring  $A^S$  of invariants of  $S = \text{End}_C A_C$  as well as  $A^{\mathcal{J}}$  where  $\mathcal{J} := \text{End}_C A_B$ , the coideal subring of  $S$ , in terms of  $A$  as the natural module over  $E := \text{End } A_B$ . The endomorphism ring  $\text{End}_E A$  is familiar from the Jacobson-Bourbaki theorem in Galois theory [7, 19].

**Theorem 5.3.** *Let  $A|B|C$  be left D3 and*

$$A^{\mathcal{J}} = \{x \in A | \forall \alpha \in \mathcal{J}, \alpha(x) = \alpha(1)x\}.$$

*Then  $A^{\mathcal{J}} \cong \text{End}_E A$  via the anti-isomorphism  $x \mapsto \rho_x$ .*

*Proof.* We first note that  $A^{\mathcal{J}} = \{x \in A | \forall f \in E, y \in A, f(yx) = f(y)x\}$ . The inclusion  $\supseteq$  easily follows from letting  $y = 1_A$  and  $\alpha \in \mathcal{J} \subseteq E$ . The reverse inclusion follows from Theorem 5.1. Since  $E \cong A \otimes_V \mathcal{J}$ , let  $f \circ \lambda_y \in E$  decompose as  $a^1 \otimes g^2 \in A \otimes_V \mathcal{J}$  for an arbitrary  $y \in A$ . Given  $x \in A$  such that  $\alpha(x) = \alpha(1)x$  for each  $\alpha \in \mathcal{J}$ , then

$$f(yx) = a^1 g^2(x) = a^1 g^2(1)x = f(y)x.$$

It follows from these considerations that  $\rho_x \in \text{End}_E A$  for  $x \in \mathcal{J}$ , since  $\rho_x(f(a)) = f(\rho_x(a))$  for each  $f \in E, a \in A$ .

Now an inverse mapping  $\text{End}_E A \rightarrow A^{\mathcal{J}}$  is given by  $G \mapsto G(1)$ . Of course  $\rho_x(1) = x$ . Note that  $G(1) \in A^{\mathcal{J}}$ , since for  $\alpha \in \mathcal{J}$ , we have  $\alpha(G(1)) = G(\alpha(1)) = \lambda_{\alpha(1)}G(1)$ , since  $\lambda_a \in E$  for all  $a \in A$ . Finally, we note that  $G(a) = G \circ \lambda_a(1) = aG(1)$ , since  $\lambda_a \in E$ , whence  $G = \rho_{G(1)}$  for each  $G \in \text{End}_E A$ .  $\square$

The following clarifies and extends part of [11, 4.1]. Let  $S$  denote the bialgebroid  $\text{End}_B A_B$  below and  $E$  as before is  $\text{End}_B A_B$ .

**Corollary 5.4.** *If  $A|B$  is left D2, then  $A^S \cong \text{End}_E A$ . Thus if  $A_B$  is balanced,  $A^S = B$ .*

*Proof.* Follows by Prop. 2.4 and from the theorem by letting  $B = C$ . We note additionally from its proof that

$$(40) \quad A^S = \{x \in A \mid \forall \alpha \in S, \alpha(x) = x\alpha(1)\}$$

since  $\rho_{\alpha(1)} \in E$  in this case.

If  $A_B$  is balanced,  $\text{End}_E A = \rho(B)$  by definition. This recovers the result in [11, Section 4].  $\square$

In other words, this corollary states that the invariant subring of  $A$  under the action of the bialgebroid  $S$  is (anti-isomorphic to) the bicommutator of the natural module  $A$ . Sugano studies the derived ring extension  $A^*|B^*$  of bicommutants of a ring extension  $A|B$ , where  $M_A$  is a faithful module,  $E := \text{End } M_A$ ,  $\mathcal{E} := \text{End } M_B$ ,  $A^* = \text{End}_E M$ ,  $B^* = \text{End}_{\mathcal{E}} M$  and there are natural monomorphisms  $A \rightarrow A^*$  and  $B \rightarrow B^*$  commuting with the mappings  $B \rightarrow A$  and  $B^* \rightarrow A^*$  [20]: in these terms,  $A^S \subseteq A$  is then the bicommutator of  $A_A$  over the depth two extension  $A|B$ .

## 6. A JACOBSON-BOURBAKI CORRESPONDENCE FOR AUGMENTED RINGS

The Jacobson-Bourbaki correspondence is usually given between subfields  $F$  of finite codimension in a field  $E$  on the one hand, and their linear endomorphism rings  $\text{End } E_F$  on the other hand. A subring of  $\text{End } E_F$  which is itself an endomorphism ring of this form is characterized by containing  $\lambda(E)$  and being finite dimensional over this. The inverse correspondence associates to such a subring  $R \subseteq \text{End } E_F$ , the subfield  $\text{End}_R E$ , since  ${}_R E$  is simple as a module. (The centralizer or commutant of  $R$  in  $\text{End } E_{\mathbb{Z}}$  in other words.) The correspondences are inverse to one another by the Jacobson-Chevalley density theorem, and may be extended to division rings by an exercise [7, Section 8.2].

Usual Galois theory follows from this correspondence, for if  $E^G = F$  where  $G$  is a finite group of automorphisms of  $E$ , then  $\text{End } E_F \cong E \# G$  and subrings of the form  $\text{End } E_K$  correspond to the subrings  $E \# H$  where  $H$  is a subgroup of  $G$  such that  $E^H = K$  for an intermediate field  $K$  of  $F \subseteq E$ . In this section, we will use a similar idea to pass from the Jacobson-Bourbaki correspondence to the correspondence  $A|B \mapsto \text{End}_B A_B$  and inverse  $S \mapsto A^S$  for certain Hopf subalgebroids  $S$  of  $\text{End}_B A_B$  for certain depth two extensions  $A|B$ . First, we will give an appropriate generalization of the Jacobson-Bourbaki correspondence to noncommutative algebra, with a proof similar to Winter [23, Section 2].

For the purposes below, we say an *augmented ring*  $(A, D)$  is a ring  $A$  with a ring homomorphism  $A \rightarrow D$  where  $D$  is a division ring. Examples are division rings, local rings, Hopf algebras and augmented algebras. A subring  $\mathcal{R}$  of  $\text{End } A := \text{End } A_{\mathbb{Z}}$  containing  $\lambda(A)$ , left finitely generated over this, where  ${}_{\mathcal{R}} A$  is simple, is said to be a *Galois subring*.

**Theorem 6.1** (Jacobson-Bourbaki correspondence for noncommutative augmented rings). *Let  $(A, D)$  be an augmented ring. There is a one-to-one correspondence between the set of division rings  $B$  within  $A$ , where  $B$  is a subring of  $A$  and  $A_B$  is a finite dimensional right vector space, and the set of Galois subrings of  $\text{End } A$ . The correspondence is given by  $B \mapsto \text{End } A_B$  with inverse correspondence  $\mathcal{R} \mapsto \text{End } \mathcal{R}A$ .*

*Proof.* We first show that if  $B$  is a division ring and subring of  $A$  of finite right codimension, then  $E = \text{End } A_B$  is a Galois subring and  $\text{End } {}_E A \cong B$ . We will need a theory of left or (dually) right vector spaces over a division ring as for example to be found in [8, chap. 4]. Suppose  $[A : B]_r = d$ .

Since  $\text{End } A_B$  is isomorphic to square matrices of order  $d$  over the division ring  $B$ , it follows that  $\text{End } A_B$  is finitely generated over the algebra  $\lambda(A)$  of left multiplications of  $A$ . Also  ${}_E A$  is simple, since  $E = \text{End } A_B$  acts transitively on  $A$ . Hence  $\text{End } {}_E A$  is a division ring. Since

$$(41) \quad A_B = B_B \oplus W_B$$

for some complementary subspace  $W$  over  $B$ , it follows from Morita's lemma ("generator modules are balanced") that in fact  $B \cong \text{End } {}_E A$ .

Conversely, let  $\mathcal{R}$  be a Galois subring. Let  $F^{\text{op}} = \text{End } \mathcal{R}A$  be the division ring (by Schur's lemma) contained in  $A^{\text{op}}$  (since  $A \subseteq \mathcal{R}$  and  $\text{End } {}_A A \cong A^{\text{op}}$ ). To finish the proof we need to show that  $[A : F]_r < \infty$  and  $\mathcal{R} = \text{End } A_F$ .

Since  $\mathcal{R}$  is finitely generated over  $A$ , we have  $s_1, \dots, s_n \in \mathcal{R}$  such that

$$\mathcal{R} = As_1 + \dots + As_n.$$

Let  $e_1, \dots, e_m \in A$  be linearly independent in the right vector space  $A$  over  $F$ . Since  $\mathcal{R}A$  is simple, the Jacobson-Chevalley density theorem ensures the existence of elements  $r_1, \dots, r_m \in \mathcal{R}$  such that for all  $i$  and  $k$ ,

$$r_i(e_k) = \delta_{ik} 1_A.$$

By the lemma below and the hypothesis that  $A$  is an augmented ring,  $m \leq n$ . With a maximal linear independent set of vectors  $e_i$  in  $A$ , we may assume  $e_1, \dots, e_m$  a basis for  $A_F$ . By definition of  $F$ , we have  $\mathcal{R} \subseteq \text{End } A_F$ . Let  $E_{ij} := e_i r_j$  for  $1 \leq i, j \leq m$  in  $\mathcal{R}$ . Since  $E_{ij}(e_k) = \delta_{jk} e_i$ , these are matrix units which span  $\text{End } A_F$ . Hence  $\text{End } A_F = \mathcal{R}$ .  $\square$

**Lemma 6.2.** *Let  $s_1, \dots, s_n \in \text{End } A_{\mathbb{Z}}$  where  $(A, D)$  is an augmented ring. Suppose that*

$$r_1, \dots, r_m \in As_1 + \dots + As_n$$

*and there are elements  $e_1, \dots, e_m \in A$  such that  $r_i(e_k) = \delta_{ik} 1_A$  for  $1 \leq i, k \leq m$ . Then  $m \leq n$ .*

*Proof.* By the hypothesis, there are elements  $a_{ij} \in A$  such that  $r_i = \sum_{j=1}^n a_{ij} s_j$  for each  $i = 1, \dots, m$ . Then for  $1 \leq i, k \leq m$ ,

$$\sum_{j=1}^n a_{ij} s_j(e_k) = r_i e_k = \delta_{ik} 1_A.$$

Applying the ring homomorphism  $A \rightarrow D$  into the division ring  $D$ , where  $a_{ij} \mapsto d_{ij}$ ,  $s_j(e_k) \mapsto z_{jk}$ , we obtain the matrix product equation,

$$\begin{pmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \vdots & \vdots \\ d_{m1} & \cdots & d_{mn} \end{pmatrix} \begin{pmatrix} z_{11} & \cdots & z_{1m} \\ \vdots & \vdots & \vdots \\ z_{n1} & \cdots & z_{nm} \end{pmatrix} = \begin{pmatrix} 1_D & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1_D \end{pmatrix}$$

This shows in several ways that  $m \leq n$ ; for example, by the rank + nullity theorem for right vector spaces [8, Ch. 4, corollary 2.4].  $\square$

Let  $A \supseteq C$  be a D2 ring extension, so that  $S := \text{End}_C A_C$  is canonically a left bialgebroid over the centralizer  $A^C$ . Any D2 subextension  $A \supseteq B$  has sub- $R$ -bialgebroid  $\mathcal{S} := \text{End}_B A_B$  where  $R = A^B \subseteq A^C$ . If all extensions are balanced, as in the situation we consider above, we recover the intermediate D2 subring  $B$  by  $\mathcal{S} \rightsquigarrow A^{\mathcal{S}} = B$ . Whence  $B \rightsquigarrow \mathcal{S}$  is a surjective correspondence and Galois connection between the set of intermediate D2 subrings of  $A \supseteq C$  and the set of sub- $R$ -bialgebroids of  $S$  where  $R$  is a subring of  $A^C$ . We will widen our perspective to include D3 intermediate subrings  $B$ , i.e. D3 towers  $A \supseteq B \supseteq C$ , and left coideal subrings of  $S$  in order to pass from surjective Galois connection to Galois correspondence.

The Galois correspondence given by  $B \rightsquigarrow \text{End}_C A_B$  and  $\mathcal{J} \rightsquigarrow A^{\mathcal{J}}$  will factor through the Jacobson-Bourbaki correspondence sketched in the theorem above. We will apply Theorems 5.3, 5.2 and 2.8 to do this. We will need a notion of *Galois left coideal subring*  $\mathcal{J}$  of a left  $V$ -bialgebroid  $S$ . For this we require of the left coideal subring  $\mathcal{J} \subseteq \text{End}_C A_C$  that

- (1) the module  ${}_V \mathcal{J}$  is finitely generated projective where  $V = A^C$ ;
- (2)  $A$  has no proper  $\mathcal{J}$ -stable left ideals.

**Theorem 6.3.** *Let  $A \supseteq C$  be a D2 extension of an augmented ring  $A$  over a division ring  $C$ , with centralizer  $A^C$  denoted by  $V$  and left  $V$ -bialgebroid  $\text{End}_C A_C$  by  $S$ . Suppose  $A_V$  is faithfully flat. Then the left D3 intermediate division rings of  $A \supseteq C$  are in Galois correspondence with the Galois left coideal subrings of  $S$ .*

*Proof.* Since  $C \subseteq A$  is D2 and left or right split (as in eq. 41), we may apply a projection  $A_C \rightarrow C_C$  to the left D2 quasibase eq. to see that  $A_C$  is a finite dimensional right vector space. For the same reasons, each extension  $A \supseteq B$  (for an intermediate division ring  $B$ ) is balanced by Morita's lemma. If  $B$  is additionally a left D3 intermediate ring, with  $\mathcal{J} = \text{End}_C A_B$  a left coideal subring of the bialgebroid  $S$  by Corollary 5.2 we have by Theorem 5.3 that the invariant subring  $A^{\mathcal{J}} = B$ . We just note that  ${}_V \mathcal{J}$  is f.g. projective by the opposite or dual of Lemma 4.3, and that a proper  $\mathcal{J}$ -stable left ideal of  $A$  would be a proper  $\text{End}_C A_B$ -stable left ideal in contradiction of the transitivity argument in Theorem 6.1. Thus  $B \mapsto \text{End}_C A_B$  is a surjective order-reversing correspondence between the set of left D3 intermediate division rings  $A \supseteq B \supseteq C$  into the set of Galois left coideal subrings of the  $V$ -bialgebroid  $S$ .

Suppose we are given a Galois left coideal subring  $\mathcal{I}$  of  $S = \text{End}_C A_C$ . Then the smash product ring  $A \rtimes \mathcal{I}$  has image we denote by  $\mathcal{R}$  in  $\text{End}_C A_C$  via  $a \otimes_V \alpha \mapsto \lambda_a \circ \alpha$  that is clearly a Galois subring, since  $\lambda(A) \subseteq \mathcal{R}$  and is a finitely generated extension; also the module  ${}_R A$  is simple by hypothesis (2) above. Then  $B = \text{End}_R A$  is an intermediate division ring between  $C \subseteq A$ , and  $\mathcal{R} = \text{End}_B A_B$  by Theorem 6.1. Since  $\mathcal{I} \hookrightarrow S$  and  ${}_V \mathcal{I}$  is flat, it follows from  $A \otimes_V S \cong \text{End}_C A_C$  that  $\text{End}_B A_B \cong A \otimes_V \mathcal{I}$  via

the mapping above. Note that  $\mathcal{I} \subseteq \text{End } A_B \cap S = \text{End } {}_C A_B$  and let  $Q$  be the cokernel. Since  $A \otimes_V \mathcal{I} \cong \mathcal{R} \cong A \otimes_V \text{End } {}_C A_B$  it follows that  $A \otimes_V Q = 0$ . Since  $A_V$  is faithfully flat,  $Q = 0$ , whence  $\mathcal{I} = \text{End } {}_C A_B$ . Finally,  $\text{End } A_B$  is isomorphic to an  $A$ - $C$ -bimodule direct summand of  $A^N$ , since  ${}_V \mathcal{I} \oplus * \cong V^N$  for some  $N$ , to which we apply the functor  ${}_A A_C \otimes_V -$ . Since  $A_B$  is finite free, it follows from Theorem 2.8 that  $A \supseteq B \supseteq C$  is left D3.  $\square$

If  $A$  or  $V$  is a division ring, the faithful flatness hypothesis in the theorem is clearly satisfied. In connection with this theorem we note the following criterion for a depth three tower of division algebras.

**Proposition 6.4.** *Suppose  $C \subseteq B \subseteq A$  is a tower of division rings where the right vector space  $A_B$  has basis  $\{a_1, \dots, a_n\}$  such that*

$$(42) \quad Ca_i \subseteq a_i B \quad (i = 1, \dots, n)$$

*Then  $A|B|C$  is left D3.*

*Proof.* It is easy to compute that  $x \otimes_B 1 = \sum_i a_i \otimes_B a_i^{-1} \beta_i(x)$  for all  $x \in A$ . Here  $\beta_i$  is the rank one projection onto the right  $B$ -span of the basis element  $a_i$  along the span of  $a_1, \dots, \hat{a}_i, \dots, a_n$ , and  $a_i^{-1} \otimes_B a_i \in (A \otimes_B A)^C$  for each  $i$ . Of course,  $\beta_i \in \text{End } {}_C A_B$ , so  $A|B$  is left D3.  $\square$

We may similarly prove that the tower is rD3 if  ${}_B A$  has basis  $\{a_i\}$  satisfying  $a_i C \subseteq Ba_i$ . When  $B = C$  we deduce the following criterion for a depth two subalgebra pair of division rings. For example, the real quaternions  $A = \mathbb{H}$ , and subring  $B = \mathbb{C}$  meet this criterion.

**Corollary 6.5.** *Suppose  $B \subseteq A$  is a subring pair of division rings where the left vector space  ${}_B A$  has basis  $\{a_1, \dots, a_n\}$  such that*

$$(43) \quad a_i B = Ba_i \quad (i = 1, \dots, n).$$

*Then  $A|B$  is depth two.*

Two remarks will close this section. First, if the centralizer  $V$  of a depth two proper extension  $A|C$  is contained in  $C$  (as in the example  $C = \mathbb{C}$  and  $A = \mathbb{H}$  just mentioned above), then  $\text{End } {}_C A_C$  is a skew Hopf algebra over the commutative base ring  $V$  [15]. Any intermediate ring  $B$  of  $A|C$ , for which  $A|B$  is D2, has skew Hopf algebra  $\text{End } {}_B A_B$  over  $R = A^B$  for the same reason, since  $R \subseteq V \subseteq C \subseteq B$ . It is interesting to determine under what conditions these are skew Hopf subalgebras, i.e., the antipodes are compatible under the sub- $R$ -bialgebroid structures.

Second, it is an intriguing possibility that the theory in this paper extends to depth  $n$  endomorphism towers over a Frobenius extension of simple algebras in a full algebraic version of the Galois theory for subfactors in Nikshych and Vainerman [18].

## 7. APPLICATION TO FIELD THEORY

Given a separable finite field extension  $F \subseteq E$  Szlachányi shows that there is a Galois connection between intermediate fields and weak Hopf subalgebras of  $\text{End } E_F$ . A *weak Hopf algebra*  $H$  the reader will recall from the already classic [3] is a weakening of the notion of Hopf algebra to include certain non-unital coproducts, non-homomorphic counits with weakened antipode equations. There are certain



canonical coideal subalgebras  $H^L$  and  $H^R$  that are separable algebras and anti-isomorphic copies of one another via the antipode. Nikshych and Etingof [5] have shown that  $H$  is a Hopf algebroid over the separable algebra  $H^L$ , and conversely the author and Szlachányi [11] have shown that Hopf algebroids over a separable algebra are weak Hopf algebras. Let's revisit one of the important, motivating examples.

**Example 7.1.** Let  $\mathcal{G}$  be a finite groupoid with  $x, y \in \mathcal{G}_{\text{obj}}$  the objects and  $g, h \in \mathcal{G}_{\text{arrows}}$  the invertible arrows (with sample elements). Let  $s(g)$  and  $t(g)$  denote the source and target objects of the arrow  $g$ . Suppose  $k$  is a field. Then the groupoid algebra  $H = k\mathcal{G}$  (defined like a quiver algebra, where  $gh = 0$  if  $t(h) \neq s(g)$ ) is a weak Hopf algebra with coproduct  $\Delta(g) = g \otimes_k g$ , counit  $\varepsilon(g) = 1$ , and antipode  $S(g) = g^{-1}$ . Since the identity is  $1_H = \sum_{x \in \mathcal{G}_{\text{obj}}} \text{id}_x$ , we see that  $\Delta(1_H) \neq 1_H \otimes 1_H$  if  $\mathcal{G}_{\text{obj}}$  has two or more objects. Notice too that  $\varepsilon(gh) \neq \varepsilon(g)\varepsilon(h)$  if  $gh = 0$ .

The Hopf algebroid structure has total algebra  $H$ , and has base algebra the separable algebra  $k\mathcal{G}_{\text{obj}}$ , which is a product algebra  $k^N$  where  $N = |\mathcal{G}_{\text{obj}}|$ . The source and target maps of the Hopf algebras  $s_L, t_L : R \rightarrow H$  are simply  $s_L = t_L : x \mapsto \text{id}_x$ . The resulting bimodule structure  ${}_R H_R = {}_{s_L, t_L} H$  is given by  $x \cdot g \cdot y = g$  if  $x = y = t(g)$ , 0 otherwise. The coproduct is  $\Delta(g) = g \otimes_R g$ , counit  $\varepsilon(g) = t(g)$ , and antipode  $S(g) = g^{-1}$ . This defines a Hopf algebroid in the sense of Lu and Xu. That this is also a Hopf algebroid in the sense of Böhm-Szlachányi may be seen by defining a right bialgebroid structure on  $H$  via the counit  $\varepsilon_r(g) = s(g)$ .

If  $\mathcal{G}$  is the finite set  $\{1, \dots, n\}$  with singleton hom-groups suggestively denoted by  $\text{Hom}(\underline{i}, \underline{j}) = \{e_{ji}\}$  for all  $1 \leq i, j \leq n$ , the groupoid algebra considered above is the full matrix algebra  $H = M_n(k)$  and  $R$  is subalgebra of diagonal matrices. Note that the projection  $\Pi^L (= \varepsilon_t$  in [5]) defined as  $\Pi^L(x) = \varepsilon(1_{(1)}x)1_{(2)}$  is given here by  $e_{ij} \mapsto e_{ii}$ . Similarly,  $\Pi^R(e_{ij}) = e_{jj}$ .

In [22], Szlachányi shows that although Hopf-Galois separable field extensions do not have a universal Hopf algebra as “Galois quantum group,” they have a universal weak Hopf algebra or “Galois quantum groupoid.” For example, the field  $E = \mathbb{Q}(\sqrt[4]{2})$  is a four dimensional separable extension of  $F = \mathbb{Q}$  which is Hopf-Galois with respect to two non-isomorphic Hopf algebras,  $H_1$  and  $H_2$  [6]. However, the endomorphism ring  $\text{End } E_F$  is then a smash product in two ways,  $E \# H_i$ ,  $i = 1, 2$ , and is a weak Hopf algebra over the separable  $F$ -algebra  $E$ . It is universal in a category of weak Hopf algebras viewed as left bialgebroids [22, Theorem 2.2], with modifications to the definition of the arrows resulting (see [22, Prop. 1.4] for the definition of weak left morphisms of weak bialgebras). The separable field extensions that are Hopf-Galois may then be viewed as being weak Hopf-Galois with a uniqueness property.

The following corollary addresses an unanswered question in [22, Section 3.3]. Namely, there is a Galois connection between intermediate fields  $K \subseteq F \subseteq E$  of a separable (finite) field extension  $E | K$  and weak Hopf subalgebras of the weak Hopf algebra  $\mathcal{A} := \text{End } E_K$  that include  $E$  as left multiplications. The correspondences are denoted by

$$\text{Sub}_{WHA/K}(\mathcal{A}) \xrightarrow{\text{Fix}} \text{Sub}_{Alg/K}(E)$$

which associates to a weak Hopf subalgebra  $W$  of  $\text{End } E_K$  the subfield

$$\text{Fix}(W) = \{x \in E | \forall \alpha \in W, \alpha(x) = \alpha(1)x\},$$

in other words,  $E^W$ , and the correspondence

$$\text{Sub}_{\text{Alg}/K}(E) \xrightarrow{\text{Gal}} \text{Sub}_{WHA/K}(\mathcal{A})$$

where the intermediate subfield  $K \subseteq F \subseteq E$  gets associated to its Galois algebra

$$\text{Gal}(F) = \{\alpha \in \mathcal{A} \mid \forall x \in E, y \in F, \alpha(xy) = \alpha(x)y\}.$$

Clearly  $\text{Gal}(F) = \text{End } E_F$ .

Szlachányi [22, 3.3] notes that  $\text{Gal}$  is a surjective correspondence, since  $F = \text{Fix}(\text{Gal}(F))$  for each intermediate subfield (e.g. since  $E_F$  is a generator module, it is balanced by Morita's lemma).  $\text{Gal}$  is indeed a one-to-one correspondence by

**Corollary 7.2.** *Gal and Fix are inverse correspondences between intermediate fields of a separable field extension  $E|K$  and weak Hopf subalgebras of the full linear endomorphism algebra  $\text{End } E_K$ .*

*Proof.* We just need to apply the Jacobson-Bourbaki correspondence with a change of notation. Before changing notation, first note that if  $A \supseteq B$  is a depth two extension where  $B$  is a commutative subring of the center of  $A$ , then the centralizer  $A^B = A$  and the left bialgebroid  $\text{End}_B A_B = \text{End } A_B$  over  $A$ . Indeed, the  $B$ -algebra  $A$  is depth two iff it is finite projective and faithfully flat as a  $B$ -module. If  $A$  and  $B$  are fields, this reduces to: depth two extension  $A|B \Leftrightarrow$  finite extension  $A|B$ . If  $A|B$  is a Frobenius extension (as are separable extensions of fields), there is an antipode on  $\text{End}_B A_B$  defined in terms of the Frobenius homomorphism (such as the trace map of a separable field extension [16]) and its dual bases [4]. Now, changing notation, we have a bialgebroid  $\text{End } E_K$  over the separable  $F$ -algebra  $E$ , or equivalently a weak bialgebra — which becomes a weak Hopf algebra via an involutive antipode given in terms of the trace map and its dual bases [22, eq. (3.5)].

Given a weak Hopf subalgebra  $W$  of  $\text{End } E_K$  containing  $\lambda(E)$ , it is automatically finite dimensional over  $E$  and  ${}_W E$  is simple since a submodule is a  $W$ -stable ideal, but  $E$  is a field. Hence,  $W$  is a Galois subring and the Theorem 6.1 shows that  $\text{End } {}_W E \cong E^W$  is an intermediate field  $F$  between  $K \subseteq E$ , such that  $\text{End } E_F = W$ . But  $\text{Gal}(F) = \text{End } E_F$  has been noted above. Hence,  $\text{Gal}(\text{Fix}(W)) = W$ .  $\square$

The only reason we need restrict ourselves to *separable* field extensions above is to acquire a fixed base algebra that is a separable algebra, so that we acquire antipodes from Frobenius extensions, and Hopf algebroids become weak Hopf algebras. Let us be clear on what happens when we drop this hypothesis. For the purpose of the next corollary, we define a sub- $R$ -bialgebroid of bialgebroid  $(H, R, s_L, t_L, \Delta, \varepsilon)$  to be a subalgebra  $V$  of the total algebra  $H$  with the same base algebra  $R$ , source  $s_L$  and target  $t_L$  maps having image within  $V$ , and  $V$  is a sub- $R$ -coring of  $(H, \Delta, \varepsilon)$ .

**Corollary 7.3.** *Let  $E \supseteq K$  be a finite field extension. Then the poset of intermediate subfields is in Galois correspondence with the poset of sub- $E$ -bialgebroids of  $\text{End } E_K$ .*

*Proof.* This follows from the Jacobson-Bourbaki correspondence, where intermediate field  $F \mapsto \text{End } E_F$  with inverse, Galois subring  $R \mapsto \text{End } {}_R E$ , with the same proof as in the previous corollary. Note from the proof of Jacobson-Bourbaki in the field context that any subring of  $\text{End } E_K$  containing  $\lambda(E)$  is indeed of the form  $\text{End } E_F$  for some intermediate  $K \subseteq F \subseteq E$ , and therefore the left bialgebroid of the depth two (= finite) field extension  $F \subseteq E$ , and sub- $E$ -bialgebroid of  $\text{End } E_K$ .  $\square$

The Jacobson-Bourbaki correspondence also exists between subfields of a finite dimensional simple algebra  $A$  and subalgebras of the linear endomorphism algebra which contain left and right multiplications [19, sect. 12.3], a theorem related to the topic of Brauer group of a field. By the same reasoning, we arrive at Galois correspondences between subfields and bialgebroids over  $A$ . Namely, let  $A^e$  denote the image of  $A \otimes_F A^{\text{op}}$  in the linear endomorphism algebra  $\text{End } A_F$  via left and right multiplication  $x \otimes y \mapsto \lambda_x \circ \rho_y$ , and  $Z(A)$  denote the center of  $A$ , which is a field since  $Z(A) \cong \text{End}_{A^e} A$ . We note that  $\text{End } A_E$  is a bialgebroid over  $A$  for any intermediate field  $F \subseteq E \subseteq Z(A)$  with Lu structure [17], and a Hopf algebroid in the special case  $E = Z(A)$  where  $A$  becomes Azumaya so  $A \otimes_E A^{\text{op}} \cong \text{End } A_E$ . The proof is quite the same as above and therefore omitted.

**Corollary 7.4.** *Let  $A$  be a simple finite dimensional  $F$ -algebra. Then the fields that are intermediate to  $F \subseteq Z(A)$  are in Galois correspondence to the sub- $A$ -bialgebroids of  $\text{End } A_F$ . In case  $A$  is a separable  $F$ -algebra, the intermediate fields are in Galois correspondence to weak Hopf subalgebras of  $\text{End } A_F$ .*

For the second part of the corollary, we note that  $A$  is separable over each intermediate field, therefore Frobenius (depth two) by a theorem of Nakayama and Eilenberg. Therefore the associated weak bialgebras have an antipode by the Larson-Sweedler-Vecsérnyes theorem.

#### REFERENCES

- [1] F. Borceaux and G. Janelidze, *Galois Theories*, C.S.A.M. **72**, Cambridge Univ. Press, 2001.
- [2] T. Brzeziński and R. Wisbauer, *Corings and Comodules*, L.M.S. **309**, Cambridge Univ. Press, 2003.
- [3] G. Böhm, F. Nill and K. Szlachányi, Weak Hopf algebras, I. Integral theory and  $C^*$ -structure, *J. Algebra* **221** (1999), 385–438.
- [4] G. Böhm and K. Szlachányi, Hopf algebroids with bijective antipodes: axioms, integrals and duals, *J. Algebra* **274** (2004), 708–750.
- [5] P. Etingof and D. Nikshych, Dynamical quantum groups at roots of 1, *Duke Math. J.* (2002).
- [6] C. Greither and B. Pareigis, Hopf Galois theory for separable field extensions, *J. Algebra* **106** (1987), 239–258.
- [7] N. Jacobson, *Basic Algebra II*, Freeman, San Francisco, 1980.
- [8] T. Hungerford, *Algebra*, Holt, Rinehart & Winston, New York, 1974.
- [9] L. Kadison and B. Külshammer, Depth two, normality and a trace ideal condition for Frobenius extensions, *Comm. Alg.* **34** (2006), 3103–3122.
- [10] L. Kadison and D. Nikshych, Hopf algebra actions on strongly separable extensions of depth two, *Adv. in Math.* **163** (2001), 258–286.
- [11] L. Kadison and K. Szlachányi, Bialgebroid actions on depth two extensions and duality, *Adv. in Math.* **179** (2003), 75–121. ArXiv Preprint RA/0108067.
- [12] L. Kadison, The endomorphism ring theorem for Galois and depth two extensions, *J. Algebra* **305** (2006), 163–184.
- [13] L. Kadison, Anchor maps and stable modules in depth two, *Appl. Categ. Struct.*, to appear. QA/0606489.
- [14] L. Kadison, Centralizers and induction, *J. Alg. & Appl.* **6** (3) (2007), 1–22. RA/0505004.
- [15] L. Kadison, Skew Hopf algebras and irreducible extensions, preprint, QA/0701427.
- [16] S. Lang, *Algebra*, ed. 3, Addison-Wesley, Paris, 1993.

- [17] J.-H. Lu, Hopf algebroids and quantum groupoids, *Int. J. Math.* **7** (1996), 47–70.
- [18] D. Nikshych and L. Vainerman, A Galois correspondence for actions of quantum groupoids on  $\text{II}_1$ -factors, *J. Func. Analysis*, **178** (2000), 113–142.
- [19] R.S. Pierce, *Associative Algebras*, G.T.M. **88**, Springer, Heidelberg, 1982.
- [20] K. Sugano, On bicommutators of modules over H-separable extension rings III, *Hokkaido Math. J.* **23** (1994), 277–289.
- [21] M.E. Sweedler, The predual theorem to the Jacobson-Bourbaki theorem, *Trans. A.M.S.* **213** (1975), 391–406.
- [22] K. Szlachányi, Galois actions by finite quantum groupoids, *Locally compact quantum groups and groupoids (Strasbourg, 2002)*, IRMA Lect. Math. Phys. **2**, de Gruyter, Berlin, 2003, 105–125. [QA/0205229](#).
- [23] D.J. Winter, A Galois theory of commutative rings, *J. Algebra* **289** (2005), 380–411.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, DAVID RITTENHOUSE LAB, 209  
 S. 33RD ST., PHILADELPHIA, PA 19104  
*E-mail address:* [lkadison@math.upenn.edu](mailto:lkadison@math.upenn.edu)